2/13/26

Return quizzes

Collect HW

Summary from last time. (Tie together things we've talked about.)

Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ is def by an $n \times m$ matrix A

Say $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} \in \mathbb{R}^m$ and $a, b \in \mathbb{R}$

Then linearity says

$$T(a\vec{v} + b\vec{w}) = aT(\vec{v}) + bT(\vec{w})$$

This is reflected in Matrix Algebra properties:

$$A\left(a\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} + b\begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}\right) = aA\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} + b\begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$$

(distributive property and scalar mult.)

§2.4 Inverses

We've talked about the inverse of a lin. transf. (if it exists):

if $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible then there exists a lin transf.

$T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$\left.\begin{aligned} (T^{-1} \circ T)(\vec{v}) &= T^{-1}(T(\vec{v})) = \vec{v} \\ (T \circ T^{-1})(\vec{v}) &= T(T^{-1}(\vec{v})) = \vec{v} \end{aligned}\right\} \forall \vec{v} \in \mathbb{R}^n$$

i.e. $T \circ T^{-1}$ and $T^{-1} \circ T$ are both the identity!

Same statement in terms of matrices: Say $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible.

If $T(\vec{v}) = A\vec{v}$ then $T^{-1}(\vec{v}) = A^{-1}\vec{v}$ where

$$AA^{-1} = A^{-1}A = I_n$$

If A^{-1} exists, A is <u>invertible</u>.

<u>Questions</u> 1. How do you tell if a given A is invertible? 2. If it is, how do you find A^{-1}?

we've started talking about this already, so quick review first.

Let A be an $n \times n$ matrix. Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$.

Let $\underset{(n\times n)}{A}\ \underset{(n\times 1)}{\vec{x}} = \vec{y} \in \mathbb{R}^n$

If A^{-1} exists then $A^{-1}(A\vec{x}) = A^{-1}\vec{y}$

$$A^{-1}(A\vec{x}) = I_n\vec{x} = \vec{x}$$

I.e. we want to solve for $\vec{x}$ in terms of $\vec{y}$. Say $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

$A\vec{x} = \vec{y}$ corresp to the lin. system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= y_1 \\ \vdots \qquad\qquad & \ \ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= y_n \end{aligned} \qquad (*)$$

We row reduce. Our goal is to get

$$x_1 = (\text{something in } y_1, \dots, y_n)$$
$$\vdots$$
$$x_n = (\quad " \quad " \quad " \quad) \qquad \text{(if possible)}$$

I.e. when we row reduce we get

$$x_1 + 0x_2 + \dots + 0x_n = (\text{something in } y_1, \dots, y_n)$$
$$0x_1 + x_2 + \dots + 0x_n = \quad " \quad " \quad "$$
$$\vdots$$
$$0x_1 + 0x_2 + \dots + x_n = \quad " \quad " \quad "$$

Theorem 2.4.3

Answers question #1

So $\boxed{A \text{ is invertible iff } RREF(A) = I_n \text{ equivalently } A \text{ is invertible iff } \operatorname{rank}(A) = n.}$

Ex Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix}$

Is A invertible? If so, find A^{-1}.

We have

$$x_1 + 2x_2 + 2x_3 = y_1 = y_1 + 0y_2 + 0y_3$$
$$3x_1 + 5x_2 + 6x_3 = y_2 = 0y_1 + y_2 + 0y_3$$
$$2x_1 + 4x_2 + 3x_3 = y_3 = 0y_1 + 0y_2 + y_3$$

But we know the variables are "place-holders" and all we need are the coefficients.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 5 & 6 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -3 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -5 & 2 & 0 \\ 0 & 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9 & 2 & 2 \\ 0 & 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} -9 & 2 & 2 \\ 3 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}}_{A^{-1}}$$

Check

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix}\begin{bmatrix} -9 & 2 & 2 \\ 3 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

they should check reverse order

$\mathrm{RREF}(A) = I_3$

so $\mathrm{rk}\, A = 3$

If $\mathrm{RREF}(A)$ had been $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ A would not be invertible.

why do we care about A^{-1}? Assume A is square, say $n \times n$.

Note 1 if A is invertible then $A\vec{x} = \vec{b}$ has a unique solution!

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$
$$\parallel$$
$$\vec{x}$$

Note 2 the linear system $A\vec{x} = \vec{0}$ always has a sol'n, whether A is invertible or not, namely $x_1 = 0$, $x_2 = 0$, ..., $x_n = 0$. The issue is whether this is the unique solution. this holds iff A is invertible

$$A\vec{x} = \vec{0} \quad \rightsquigarrow \quad \vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{0} = \vec{0}$$